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by
ROKAYA AL-AYAT
and
ROLF FÄRE

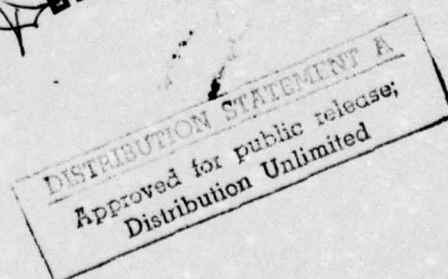
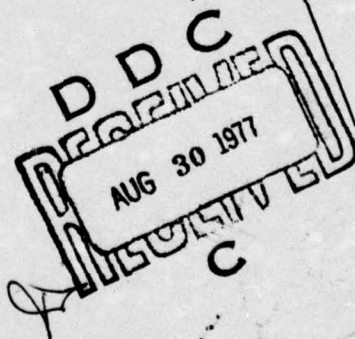
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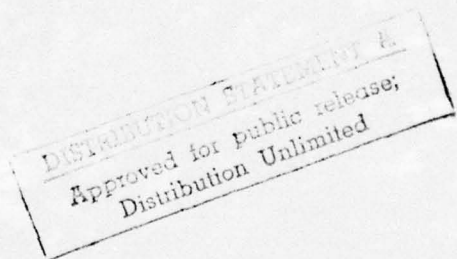
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ALMOST RAY-HOMOTHETIC PRODUCTION CORRESPONDENCES[†]

by

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ABSTRACT

A class of production correspondences for which nonproportional scaling of inputs (outputs) result in fixed scaling of outputs (inputs) is introduced. Such correspondences termed, almost ray-homothetic contain the ray-homothetic structure as a special case. Although more general this scaling law maintains the linearity of expansion paths, and under special conditions on the input (output) set it is shown that linear expansion under nonlinear change of inputs (outputs) imply almost ray-homothetic output (input) structure.

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1. INTRODUCTION

Production correspondences exhibiting certain scaling laws have been investigated over the years. Apart from the simple homogenous technologies, Shephard in [8], [9] introduced and studied semi-homogenous and homothetic structures and Eichhorn (see [1], [2], [3]) developed the class of quasi-homogenous production correspondences. It was shown in [5], that these various structures can be generated as special classes of the family of ray-homothetic structure, which in turn was characterized in terms of linear expansion paths. One should recall, that expansion paths are determined by the support vectors for benefit maximization (cost minimization) under fixed output (input) prices and proportional changes in inputs (outputs).

In this paper the more general class of almost ray-homothetic input and output structures will be introduced. In contrast to the above scaling, nonproportional changes in the inputs (outputs) will be allowed. It will be shown, however, that while generalizing the ray-homothetic class, the linearity of output (input) expansion path for almost ray-homothetic will still be preserved under nonlinear expansion of inputs (outputs).

The arguments to follow in this paper are carried out within the framework for a production technology introduced in [9]. A mapping

$x \rightarrow P(x) \subset \mathbb{R}_+^m$, of input vectors $x \in \mathbb{R}_+^n$ to subsets $P(x)$ of all output vectors $u \in \mathbb{R}_+^m$ obtainable by x is called an output correspondence.

Inversely, the input correspondence $u \rightarrow L(u) := \{x \mid u \in P(x)\}$ determines the set of all input vectors yielding the output vector $u \in \mathbb{R}_+^m$. Both $L(u)$ and $P(x)$ are assumed to satisfy the inversely related set of weak axioms in [9]. Unless specifically indicated, free disposability of inputs or outputs together with convexity of $L(u)$ or $P(x)$ are not enforced.

2. ALMOST RAY-HOMOTHETICITY

Ray-homothetic production structures were introduced in [5] to model technologies for which proportional changes in inputs (outputs) result in a fixed scaling for each output (input) mix $\frac{u}{|u|} \left(\frac{x}{|x|} \right)$. An extension of this class of technologies is introduced here to include those for which nonproportional scaling of inputs (outputs) scale outputs (inputs) as above. For this reason the output correspondence $x \rightarrow P(x)$ is defined to be almost ray-homothetic by:

Definition 1:

The output correspondence $x \rightarrow P(x)$ is called Almost Ray-Homothetic if and only if

$$(1) \quad P\left(\lambda^{\alpha_1} \cdot x_1, \lambda^{\alpha_2} \cdot x_2, \dots, \lambda^{\alpha_n} \cdot x_n\right) = \frac{F\left(\lambda^{\alpha_1} \cdot x_1, \lambda^{\alpha_2} \cdot x_2, \dots, \lambda^{\alpha_n} \cdot x_n\right)}{F(x)} \cdot P(x)$$

for $\lambda > 0$, $\alpha_i > 0$ ($i = 1, 2, \dots, n$) where $F(x)$ is nonnegative scalar valued function compatible with $P(x)$.

In the special case of $\alpha_i = \alpha > 0$ ($i = 1, 2, \dots, n$), almost ray-homotheticity becomes (simply) ray-homotheticity, i.e.,

$$P(\mu \cdot x) = \frac{F(\mu \cdot x)}{F(x)} \cdot P(x), \quad \mu > 0$$

or equivalently,

$$P(x) = \frac{F(x)}{F\left(\frac{x}{|x|}\right)} \cdot P\left(\frac{x}{|x|}\right)$$

where $|x|$ denotes the norm of $x \in \mathbb{R}_+^n$.

The input correspondence $L(u) := \{x \mid u \in P(x)\}$ is similarly defined to be almost ray-homothetic by:

Definition 2:

The input correspondence $u \rightarrow L(u)$ is Almost Ray-Homothetic if and only if

$$(2) \quad L\left(\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m\right) = \frac{G\left(\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m\right)}{G(u)} \cdot L(u)$$

for $\theta > 0$, $\beta_i > 0$ ($i = 1, 2, \dots, m$) where $G(u)$ is a scalar valued function compatible with $u \rightarrow L(u)$.

Clearly, if $\beta_i = \beta > 0$ ($i = 1, 2, \dots, m$), (2) reduces to a ray-homothetic input correspondence,

$$L(u) = \frac{G(u)}{G\left(\frac{u}{|u|}\right)} \cdot L\left(\frac{u}{|u|}\right).$$

To elucidate the nature of almost ray-homothetic correspondences, a simple example of such structure is next given. Let $x = (x_1, x_2)$, $u = (u_1, u_2)$, consider

$$(3) \quad P(x) = \phi(x) \cdot e^{\gamma \cdot \phi(x)} \cdot \left\{ (u_1, u_2) \mid 0 \leq u_1 \leq \frac{x_1^{-\alpha_1}}{x_2^{-\alpha_2}}, 0 \leq u_2 \leq \frac{x_2^{-\alpha_2}}{x_1^{-\alpha_1}} \right\}$$

for $x_i > 0$ ($i = 1, 2$) and $P(0) = \{0\}$ and where $\phi(x) = \left(\delta_1 \cdot x_1^{-\alpha_1} + \delta_2 \cdot x_2^{-\alpha_2} \right)^{-1/\alpha}$ is a Mukerji production function (see [6], [7]) and $\gamma \geq 0$.

It is easy to see that (3) is an almost ray-homothetic but not (simple) ray-homothetic output correspondence.

If the output correspondence is almost ray-homothetic, then clearly,

$$P\left(\lambda^{\alpha_1} \cdot x_1, \lambda^{\alpha_2} \cdot x_2, \dots, \lambda^{\alpha_n} \cdot x_n\right) = \Gamma(\lambda, x) \cdot P(x)$$
 , where $\Gamma(\lambda, x)$ is a given function compatible with this structure, suggesting:

Proposition 1:

The output correspondence is almost ray-homothetic if and only if

$$(4) \quad P\left(\lambda^{\alpha_1} \cdot x_1, \lambda^{\alpha_2} \cdot x_2, \dots, \lambda^{\alpha_n} \cdot x_n\right) = \Gamma(\lambda, x) \cdot P(x)$$

where $\Gamma(\lambda, x)$ is a scalar valued function compatible with the output correspondence.

Proof:

It is sufficient to prove that (4) implies (1). From (4) with $\lambda, \mu > 0$,

$$\begin{aligned} & P\left((\lambda \cdot \mu)^{\alpha_1} \cdot x_1, (\lambda \cdot \mu)^{\alpha_2} \cdot x_2, \dots, (\lambda \cdot \mu)^{\alpha_n} \cdot x_n\right) = \Gamma(\lambda \cdot \mu, x) \cdot P(x) \\ & = \Gamma\left(\lambda, \mu^{\alpha_1} \cdot x_1, \mu^{\alpha_2} \cdot x_2, \dots, \mu^{\alpha_n} \cdot x_n\right) \cdot P\left(\mu^{\alpha_1} \cdot x_1, \mu^{\alpha_2} \cdot x_2, \dots, \mu^{\alpha_n} \cdot x_n\right) \\ & = \Gamma\left(\lambda, \mu^{\alpha_1} \cdot x_1, \mu^{\alpha_2} \cdot x_2, \dots, \mu^{\alpha_n} \cdot x_n\right) \cdot \Gamma(\mu, x) \cdot P(x) , \end{aligned}$$

implying that $\Gamma(\lambda, x)$ satisfies the functional equation

$$(5) \quad \Gamma(\lambda \cdot \mu, x) = \Gamma\left(\lambda, \mu^{\alpha_1} \cdot x_1, \mu^{\alpha_2} \cdot x_2, \dots, \mu^{\alpha_n} \cdot x_n\right) \cdot \Gamma(\mu, x) .$$

To find the general solution of (5), write $x_i = \left(\frac{1}{x_i^{\alpha_i}}\right)^{\alpha_i}$ ($i = 1, 2, \dots, n$) and define $y_i = x_i^{1/\alpha_i}$. Furthermore, take $\lambda = (\mu \cdot |y|)^{-1}$, and (5) becomes

$$(6) \quad \Gamma\left(\frac{1}{|y|}, y_1^{\alpha_1}, y_2^{\alpha_2}, \dots, y_n^{\alpha_n}\right) = \Gamma\left(\frac{1}{|\mu \cdot y|}, (\mu \cdot y_1)^{\alpha_1}, (\mu \cdot y_2)^{\alpha_2}, \dots, (\mu \cdot y_n)^{\alpha_n}\right) \\ \cdot \Gamma\left(\mu, y_1^{\alpha_1}, y_2^{\alpha_2}, \dots, y_n^{\alpha_n}\right).$$

Now, define $F(y_1^{\alpha_1}, y_2^{\alpha_2}, \dots, y_n^{\alpha_n}) := \left[\Gamma\left(\frac{1}{|y|}, y_1^{\alpha_1}, y_2^{\alpha_2}, \dots, y_n^{\alpha_n}\right)\right]^{-1}$,

then by (6),

$$\Gamma(\mu, y_1^{\alpha_1}, y_2^{\alpha_2}, \dots, y_n^{\alpha_n}) = \frac{F((\mu \cdot y_1)^{\alpha_1}, (\mu \cdot y_2)^{\alpha_2}, \dots, (\mu \cdot y_n)^{\alpha_n})}{F(y_1^{\alpha_1}, y_2^{\alpha_2}, \dots, y_n^{\alpha_n})}$$

or for $y_i^{\alpha_i} = x_i$ ($i = 1, 2, \dots, n$)

$$(7) \quad \Gamma(\mu, x_1, x_2, \dots, x_n) = \frac{F(\mu^{\alpha_1} \cdot x_1, \mu^{\alpha_2} \cdot x_2, \dots, \mu^{\alpha_n} \cdot x_n)}{F(x_1, x_2, \dots, x_n)}.$$

Moreover, (7) solves the functional equation (5). Hence the proposition is proved. Q.E.D.

If the function F is almost homogeneous i.e.,

$$F(\lambda^{\alpha_1} \cdot x_1, \lambda^{\alpha_2} \cdot x_2, \dots, \lambda^{\alpha_n} \cdot x_n) = \lambda^{\alpha_0} \cdot F(x)$$

then the output correspondence is also almost homogeneous, i.e.,

$$P\left(\lambda^{\alpha_1} \cdot x_1, \lambda^{\alpha_2} \cdot x_2, \dots, \lambda^{\alpha_n} \cdot x_n\right) = \lambda^{\alpha_0} \cdot P(x) .$$

Next, consider the functional equation (4) with $\Gamma(\lambda, x)$ replaced by $\sum \left(\lambda^{\alpha_0}, \phi(x) \right)$, where ϕ is an almost homogeneous scalar valued production function, $\alpha_0 > 0$ and \sum is compatible with the output structure. In this case it will be shown that

$$(8) \quad P\left(\lambda^{\alpha_1} \cdot x_1, \lambda^{\alpha_2} \cdot x_2, \dots, \lambda^{\alpha_n} \cdot x_n\right) = \frac{F\left(\lambda^{\alpha_0} \cdot \phi(x)\right)}{F(\phi(x))} \cdot P(x) .$$

Note that for $P(x) = [0, F(\phi(x))] \subset \mathbb{R}_+$, the scalar valued production function resulting from (8), $H(x) := \max \{u \mid u \in [0, F(\phi(x))]\}$, is then almost homothetic.

Proposition 2:

The output correspondence is of the form (8) if and only if

$$(9) \quad P\left(\lambda^{\alpha_1} \cdot x_1, \lambda^{\alpha_2} \cdot x_2, \dots, \lambda^{\alpha_n} \cdot x_n\right) = \sum \left(\lambda^{\alpha_0}, \phi(x) \right) \cdot P(x)$$

where $\phi\left(\lambda^{\alpha_1} \cdot x_1, \lambda^{\alpha_2} \cdot x_2, \dots, \lambda^{\alpha_n} \cdot x_n\right) = \lambda^{\alpha_0} \cdot \phi(x)$ is an almost homogeneous scalar valued production function and \sum is compatible with $x \rightarrow P(x)$.

Proof:

It is clearly sufficient to prove that (9) implies (8). For $\lambda, \mu > 0$ using similar argument to that of Proposition 1,

$$(10) \quad \sum \left((\lambda \cdot \mu)^{\alpha_0}, \phi(x) \right) = \sum \left(\lambda^{\alpha_0}, \mu^{\alpha_0} \cdot \phi(x) \right) \cdot \sum \left(\mu^{\alpha_0}, \phi(x) \right) .$$

Define the function $F(\alpha) := \sum (\alpha, 1)$ and take $\mu^{\alpha_0} = 1$, $\phi = \mu^{\alpha_0}$.

From (10) it then follows that

$$(11) \quad \sum \left(\lambda^{\alpha_0}, \phi(x) \right) = \frac{F(\lambda^{\alpha_0} \cdot \phi(x))}{F(\phi(x))} .$$

Observing that (11) solves (10), the proposition is then proved. Q.E.D.

From the above discussion of the output structure and from the definition of almost ray-homothetic input structure it is clear that $u \rightarrow L(u)$ is almost ray-homothetic if and only if

$$(12) \quad L \left(\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m \right) = \Delta(\theta, u) \cdot L(u)$$

where $\Delta(\theta, u)$ is a scalar valued function compatible with $u \rightarrow L(u)$,

$\beta_i > 0$ ($i = 1, 2, \dots, m$) and $\theta > 0$.

3. EXPANSION PATHS

In [5] properties of expansion paths in the input (output) space resulting from proportional expansions of outputs (inputs) under fixed prices were discussed. Especially, it was shown that under the strong axioms ray-homothetic input (output) structure is a necessary and sufficient condition for linear inputs (outputs) expansion paths.

Here, nonlinear expansions of inputs (outputs) resulting in linear output (input) expansion paths are examined under almost ray-homothetic output (input) structure. For this reason, define the cost minimization set $C(u,p)$ for input prices $p \geq 0$ (i.e., $p \geq 0$ but $p \neq 0$) and output $u \geq 0$ with $L(u)$ nonempty by

$$C(u,p) = \{x \mid x \in L(u), p \cdot x = Q(u,p)\} ,$$

where $Q(u,p)$ is the cost function given by

$$Q(u,p) = \min \{p \cdot x \mid x \in L(u)\} .$$

With these notions, nonlinearly induced linear input expansion paths may now be defined.

Definition 3:

Given $p \geq 0$ and $u \geq 0$ with $L(u)$ not empty, the $(\beta_1, \beta_2, \dots, \beta_m)$ - nonlinear output expansion has linear input expansion paths if and only

if there exists a scalar valued function $\Delta(\theta, u) > 0$ such that

$$C\left(\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m, p\right) = \Delta(\theta, u) \cdot C(u_1, u_2, \dots, u_m, p) ,$$

$\theta > 0$.

The relationship between the above nonlinearly induced expansion paths and almost ray-homothetic input structures is next explored. For this input structure, the cost function will satisfy

$$Q\left(\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m, p\right) = \min \left\{ px \mid x \in \frac{G\left(\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m\right)}{G(u)} \cdot L(u) \right\}$$

$$= \frac{G\left(\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m\right)}{G(u)} \cdot Q(u, p)$$

for $\theta > 0$ $p \geq 0$ and $u \geq 0$ with $L(u)$ nonempty. Hence the cost minimization set is

$$C\left(\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m, p\right) = \frac{G\left(\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m\right)}{G(u)} \cdot C(u, p)$$

and thus the $(\beta_1, \beta_2, \dots, \beta_m)$ - nonlinear output expansion for an almost ray-homothetic input structure implies linear input expansion paths.

For the converse to hold, further conditions on the input structure $L(u)$, are imposed; namely, convexity and free disposability of inputs (i.e., $x' \geq x \in L(u) \Rightarrow x' \in L(u)$). The following lemma proved in [5] is of use.

Lemma 1:

If $L(u)$ is convex for $u \in \mathbb{R}_+^m$ and inputs are freely disposable, then $L(u) = \bigcup_{p \geq 0} C(u, p) + \mathbb{R}_+^n$.

Now, assume that the $(\beta_1, \beta_2, \dots, \beta_m)$ - nonlinear output expansion has linear input expansions i.e.,

$$C\left(\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m, p\right) = \Delta(\theta, u) \cdot C(u, p),$$

for $\theta > 0$, $\beta_i > 0$ ($i = 1, 2, \dots, m$) $u \geq 0$ and $p \geq 0$. Since $\Delta(\theta, u)$ is independent of p , it then follows

$$\bigcup_{p \geq 0} C\left(\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m, p\right) = \Delta(\theta, u) \cdot \bigcup_{p \geq 0} C(u, p)$$

by adding \mathbb{R}_+^n to both sides of the above expression and using Lemma 1,

$$L\left(\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m\right) = \Delta(\theta, u) \cdot L(u).$$

Hence by arguments similar to those of Proposition 1, the following is true:

Proposition 3:

Let $L(u)$ be convex for $u \in \mathbb{R}_+^m$ and the inputs be strongly disposable, then the $(\beta_1, \beta_2, \dots, \beta_n)$ - nonlinear output expansion has linear input expansion paths if and only if the input structure is almost ray-homothetic.

For the output structure, introduce

Definition 4:

Let the output prices $r \geq 0$, the inputs $x \geq 0$ with $P(x) \neq \{0\}$, then the $(\alpha_1, \alpha_2, \dots, \alpha_n)$ - nonlinear input expansion has linear output expansion paths if and only if there is a scalar valued function $\Gamma(\lambda, x) > 0$ such that $B\left(\lambda^{\alpha_1} \cdot x_1, \lambda^{\alpha_2} \cdot x_2, \dots, \lambda^{\alpha_n} \cdot x_n, r\right) = \Gamma(\lambda, x) \cdot B(x_1, x_2, \dots, x_n, r)$, $\lambda > 0$, where $B(x, r) = \{u \mid u \in P(x), r \cdot u = R(x, r)\}$ and $R(x, r) = \max \{r \cdot u \mid u \in P(x)\}$.

Observe that under $P(x)$ convex $x \in \mathbb{R}_+^n$ and output freely disposable (i.e., $u \in P(x) \Rightarrow \{v \mid 0 \leq v \leq u\} \subset P(x)$) it can be proved that (see [5]),

$$P(x) = \left(\bigcup_{r \geq 0} B(x, r) + \mathbb{R}_-^m \right) \cap \mathbb{R}_+^m,$$

and the following proposition is clear.

Proposition 4:

Let $P(x)$ be convex for $x \in \mathbb{R}_+^n$ and the outputs be strongly disposable, then the $(\alpha_1, \alpha_2, \dots, \alpha_n)$ - nonlinear input expansion has linear output expansion paths if and only if the output structure is almost ray-homothetic.

4. INVERSELY RELATED ALMOST RAY-HOMOTHETIC STRUCTURES

New insights into production structures have been gained by assuming that both the input and the output correspondences obey simultaneously a certain scaling law. For example, it can easily be shown that if $L(u)$ and $P(x)$ are homogeneous of degrees α and β , respectively, then $\alpha \cdot \beta = 1$. If the input structure is semi-homogeneous;

$$L(\mu \cdot u) = \mu^{g\left(\frac{u}{|u|}\right)} \cdot L(u), \quad \mu > 0, \quad g\left(\frac{u}{|u|}\right) > 0$$

together with a semi-homogeneous output structure;

$$P(\lambda \cdot x) = \lambda^{h\left(\frac{x}{|x|}\right)} \cdot P(x), \quad \lambda > 0, \quad h\left(\frac{x}{|x|}\right) > 0,$$

then $h\left(\frac{x}{|x|}\right) \cdot g\left(\frac{u}{|u|}\right) = 1$ (see [9]). In [5] it was shown that for simultaneous quasi-homogeneity;

$$P(x) = k(|x|) \cdot P\left(\frac{x}{|x|}\right) \quad \text{and} \quad L(u) = \ell(|u|) \cdot L\left(\frac{u}{|u|}\right)$$

both structures have to be homogeneous. It was also concluded in [2] and [4] that for simultaneous ray-homothetic input and output structures both $P(x)$ and $L(u)$ have to be semi-homogeneous.

No simple result however can be obtained if both the input and the output structures are almost ray-homothetic. But the following relations hold

$$(13) \quad \Gamma\left(\lambda, \frac{x}{\Delta(\theta, u)}\right) \cdot P\left(\frac{x}{\Delta(\theta, u)}\right) = \Gamma(\lambda, x) \cdot P\left(\frac{x}{\Delta\left(\theta, \frac{u}{\Gamma(\lambda, x)}\right)}\right)$$

$$(14) \quad \Delta\left(\theta, \frac{u}{\Gamma(\lambda, x)}\right) \cdot L\left(\frac{u}{\Gamma(\lambda, x)}\right) = \Delta(\theta, u) \cdot L\left(\frac{u}{\Gamma\left(\lambda, \frac{x}{\Delta(\theta, u)}\right)}\right).$$

To prove this let $\left(\lambda^{\alpha_1} \cdot x_1, \lambda^{\alpha_2} \cdot x_2, \dots, \lambda^{\alpha_n} \cdot x_n\right) \in L\left(\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m\right)$. Then by almost ray-homotheticity of input and output structures;

$$\begin{aligned} \left(\lambda^{\alpha_1} \cdot x_1, \lambda^{\alpha_2} \cdot x_2, \dots, \lambda^{\alpha_n} \cdot x_n\right) &\in \Delta(\theta, u) \cdot L(u) \Leftrightarrow \\ u &\in P\left(\frac{\lambda^{\alpha_1} \cdot x_1, \lambda^{\alpha_2} \cdot x_2, \dots, \lambda^{\alpha_n} \cdot x_n}{\Delta(\theta, u)}\right) \end{aligned}$$

which is equivalent to

$$(15) \quad u \in \Gamma\left(\lambda, \frac{x}{\Delta(\theta, u)}\right) \cdot P\left(\frac{x}{\Delta(\theta, u)}\right).$$

Also,

$$\begin{aligned} \left(\lambda^{\alpha_1} \cdot x_1, \lambda^{\alpha_2} \cdot x_2, \dots, \lambda^{\alpha_n} \cdot x_n\right) &\in L\left(\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m\right) \Leftrightarrow \\ \left(\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m\right) &\in P\left(\lambda^{\alpha_1} \cdot x_1, \lambda^{\alpha_2} \cdot x_2, \dots, \lambda^{\alpha_n} \cdot x_n\right) \Leftrightarrow \\ x &\in L\left(\frac{\theta^{\beta_1} \cdot u_1, \theta^{\beta_2} \cdot u_2, \dots, \theta^{\beta_m} \cdot u_m}{\Gamma(\lambda, x)}\right) = \Delta\left(\theta, \frac{u}{\Gamma(\lambda, x)}\right) \cdot L\left(\frac{u}{\Gamma(\lambda, x)}\right). \end{aligned}$$

Equivalently,

$$(16) \quad u \in \Gamma(\lambda, x) \cdot P\left(\frac{x}{\Delta\left(\theta, \frac{u}{\Gamma(\lambda, x)}\right)}\right).$$

From relations (15) and (16), (13) follows. A similar argument apply to show that (14) also holds.

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